On the structure of the twisted Grassmann graphs(Algebraic combinatorics and the related areas of research)

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On the structure of the twisted Grassmann graphs

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1 Introduction

A graph Γ with diameter d is said to be *distance-regular* if there are integers b_i ($i = 0, \cdots, d-1$) and c_i ($i = 1, \cdots, d$) such that for any two veritices x and y such that $d(x,y) = i$,

$$\begin{aligned} b_i &= \#\{z \mid z \text{ :vertex, } d(x,z) = i+1, d(y,z) = 1\}, \\ c_i &= \#\{z \mid z \text{ :vertex, } d(x,z) = i-1, d(y,z) = 1\}. \end{aligned}$$

Let q be a prime power and n, e be integers such that $n/2 \geq e \geq 2$. The Grassmann graph $J_q(n,e)$ is a graph on the e-dimensional subspaces in an n-dimensional vector space over the finite field $GF(q)$ where two e-dimensional subspaces are adjacent if and only if they intersect in a $(e-1)$-dimensional subspace. The Grassmann graph $J_q(n,e)$ is a distance-regular graph whose parameters are

$$b_i = q^{2i+1} \begin{bmatrix} e-i \\ 1 \end{bmatrix} \begin{bmatrix} n-e-i \\ 1 \end{bmatrix}, \quad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}^2.$$

where $\begin{bmatrix} m \\ 1 \end{bmatrix} = q^{m-1} + \cdots + q + 1$.

The twisted Grassmann graphs $\tilde{J}_q(2e+1, e)$, which is constructed by E. van Dam and J. Koolen [1], is defined as follows: let H be a hyperplane of the $(2e+1)$-dimensional vector space V over $GF(q)$. Put

$$\begin{aligned} \mathcal{B}_1 &= \{W \text{ :subspace of } V \mid \dim W = e+1, W \not\subseteq H\}, \\ \mathcal{B}_2 &= \{W \text{ :subspace of } H \mid \dim W = e-1\}. \end{aligned}$$

The vertex set of $\tilde{J}_q(2e+1, e)$ is $\mathcal{B}_1 \cup \mathcal{B}_2$ and the adjacency is defined as follows: for $W_1, W_2 \in \mathcal{B}_1 \cup \mathcal{B}_2$,

$$W_1 \sim W_2 \text{ if and only if } \begin{cases} \dim(W_1 \cap W_2) = e & \text{if } W_1, W_2 \in \mathcal{B}_1, \\ \dim(W_1 \cap W_2) = e-2 & \text{if } W_1, W_2 \in \mathcal{B}_2, \\ \dim(W_1 \cap W_2) = e-1 & \text{otherwise .} \end{cases}$$

Theorem 1 [1] *The twisted Grassmann graph $\tilde{J}_q(2e+1,e)$ is distance-regular and its parameters are same as the Grassmann graph $J_q(2e+1,e)$. Moreover the automorphism group of the twisted Grassmann graph acts on the vertex set with two orbits $\mathcal{B}_1$ and $\mathcal{B}_2$.*

M. Tagami determined the automorphism group of $\tilde{J}_q(2e+1,e)$ and later J. Koolen showed another proof of the coincidence (see [3]).

Theorem 2 *The automorphism group of $\tilde{J}_q(2e+1,e)$ is just $P\Gamma L(2e+1,q)_H$.*

Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme. Suppose that $\mathcal{X}$ is Q-polynomial. For $i \in \{0,\cdots,d\}$, let A_i be a matrix indexed by X defined as follows: for two vertices x, y,

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{if } (x,y) \notin R_i, \end{cases}$$

Fix a vertex x. For $i \in \{0,\cdots,d\}$, let $E_i^* = E_i^*(x)$ be a diagonal matrix indexed by the vertex set of Γ defined by, for each vertex y,

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } d(x,y) = i, \\ 0 & \text{otherwise} \end{cases}$$

The algebra $T = T(x)$ generated by $A_0,\cdots,A_d$ and $E_0^*,\cdots,E_d^*$ over the complex field is called the *Terwilliger algebra with respect to x*. For an irreducible T-module W, if for any $i \in \{0,\cdots,d\}$, $\dim(E_i^*W) \le 1$, we say that W is *thin*, and if any irreducible T-module is thin, we say T is *thin*. Every Terwilliger algebra T has a thin module $T\mathbf{1}$ where $\mathbf{1}$ is an all-one vector. This module satisfies $\dim(E_i^*W) = 1$ for any i. If an irreducible T-module W has an integer j of $\{0,\cdots,d\}$ such that $\dim(E_i^*W) = 0$ for any $i < j$ and $\dim(E_j^*W) \neq 0$, we say that W is *of endpoint j* (ref. [4].) P. Terwilliger conjectured the following: If a commutative association scheme $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ is Q-polynomial, then one of the following holds (1) $\mathcal{X}$ is formally self-dual or (2) for any $x \in X$, the Terwilliger algebra $T(x)$ is thin.

It is well-known that the association scheme obtained from the Grassmann graph is Q-polynomial. The association scheme is not formally self-dual but for any $x \in X$, the Terwilliger algebra $T(x)$ is thin, that is, the above conjecture holds. Since conditions of Q-polynomial and self-dual depends only on the parameters b_i and c_i, the association scheme obtained from the twisted Grassmann graph is also Q-polynomial but not formally self-dual.

For a graph Γ, let $A = A_\Gamma$ be the adjacency matrix of Γ. We call the eigenvalues and multiplicities of A the *eigenvalues* and *multiplicties* of Γ respectively. Let $\theta_1, \theta_2, \cdots, \theta_t$ and $m_1, m_2, \cdots, m_t$ are respectively the eigenvalues and corresponding multiplicities of Γ. Then for any non-negative integer i,

$$\sum_{p=1}^{t} m_p \theta_p^i = \text{Trace } A^i = \#\{ \text{ closed path of length } i \text{ in } \Gamma\}$$

where closed path of length i means that a sequence $x_1, x_2, \cdots, x_i$ of vertices satisfying any consective two vertices are adjacent and x_i and x_1 is also adjacent. Define that a closed path of length 0 is a vertex. For a distance-regular graph Γ with parameters b_i and c_i, let A be a adjacency matrix of the local graph with respect to a vertex x. We can easily see that

$$\text{Trace } A^0 = b_0, \text{ Trace } A^1 = 0, \text{ and Trace } A^2 = b_0(b_0 - b_1 - 1).$$

In particular, for the Grassmann graph $J_q(2e+1, e)$ and the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$, we have that Trace $A^0 = q\begin{bmatrix} e \\ 1 \end{bmatrix}\begin{bmatrix} e+1 \\ 1 \end{bmatrix}$, Trace $A^1 = 0$ and Trace $A^2 = q\begin{bmatrix} e \\ 1 \end{bmatrix}\begin{bmatrix} e+1 \\ 1 \end{bmatrix}\left(q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1\right)$.

A distance-regular graph Γ has *classic parameter* (d, q, α, β) if

$$\begin{aligned} b_i &= \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) \\ c_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \end{aligned}$$

The Grassmann graph $J_q(n, e)$ has classic parameter $\left(e, q, q, q\begin{bmatrix} n-e \\ 1 \end{bmatrix}\right)$. Similarly the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$ also has calssic parameter $\left(e, q, q, q\begin{bmatrix} e+1 \\ 1 \end{bmatrix}\right)$.

2 Computing the eigenvalues of graphs

In this section, we show the method to compute the eigenvalues of graphs. For a graph Γ on the vertex set V and for an automorphism group G, not necessarily Aut(G), consider actions of G on V and $V \times V$. Let $O_1, O_2, \cdots O_p$ be the orbits on V and $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_{p'}$ be the orbits on $V \times V$. Suppose that $\mathcal{O}_1, \cdots, \mathcal{O}_s$ satisfies that $\cup_{i=1}^{s}\mathcal{O}_i = O_1 \times V$. For $1 \le i, j \le s$ and for $(x, y) \in \mathcal{O}_i$,

$$P(i,j) = \#\{z : \text{ vertex } \mid (x,z) \in \mathcal{O}_j \text{ and } y, z \text{ are adjacent}\}$$

is independent of the choice of (x, y) and only depends on i and j. Let $P_1 = (P(i,j))_{1 \le i,j \le s}$. Similarly we can construct matrices $P_2, \cdots, P_p$.

Proposition 3 *The union of eigenvalues of P_i's is just the eigenvalues of Γ.*

From now on, we put Γ as the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$. We consider an action of the stabilizer of $U \in \mathcal{B}_1 \cup \mathcal{B}_2$ in $G = P\Gamma L(2e+1, q)_H$ as automorphism group. Then we need to separate computation of eigenvalues in each case $U \in \mathcal{B}_1$ or $U \in \mathcal{B}_2$.

2.1

Fix $U \in \mathcal{B}_1$. Then the neighbors of U in Γ consists of the following two sets:

$$A := \{W \in \mathcal{B}_1 | \ W \text{ is adjacent to } U\}, \ B := \{W \in \mathcal{B}_2 | \ W \text{ is adjacent to } U\}.$$

The A and B forms the G_U-orbitals on the neighbors of U and G_U-orbitals on $A \cup B$ are following:

$$\begin{aligned}
A_0 &:= \{(W_1, W_1) \mid W_1 \in A\}, \\
A_1 &:= \{(W_1, W_2) \in A \times A \mid W_1 \cap U = W_2 \cap U, \langle W_1, U\rangle = \langle W_2, U\rangle, W_1 \neq W_2\}, \\
A_2 &:= \{(W_1, W_2) \in A \times A \mid W_1 \cap U = W_2 \cap U, \langle W_1, U\rangle \neq \langle W_2, U\rangle\}, \\
&\quad \text{(In the cases of } A_1 \text{ and } A_2,\ W_1 \cap W_2 \text{ is a } (e-2)\text{-dimensional subspace in } U.) \\
A_3 &:= \{(W_1, W_2) \in A \times A \mid W_1 \cap W_2 : (e-2)\text{-dimensional subspace not in } U\}, \\
A_4 &:= \{(W_1, W_2) \in A \times A \mid W_1 \cap W_2 : (e-3)\text{-dimensional subspace}\}, \\
&\quad \text{(In this case, } W_1 \cap W_2 \text{ is in } U.) \\
AB_1 &:= \{(W_1, W_2) \in A \times B \mid W_1 \subset W_2\}, \\
AB_2 &:= \{(W_1, W_2) \in A \times B \mid W_1 \not\subset W_2\}, \\
BA_1 &:= \{(W_1, W_2) \in B \times A \mid W_2 \subset W_1\} (= \text{transpose of } AB_1), \\
BA_2 &:= \{(W_1, W_2) \in B \times A \mid W_2 \not\subset W_1\} (= \text{transpose of } AB_2), \\
B_0 &:= \{(W_1, W_1) \mid W_1 \in B\}, \\
B_1 &:= \{(W_1, W_2) \in B \times B \mid W_1 \cap W_2 :\ e\text{-dimensional subspace in } H\} \\
B_2 &:= \{(W_1, W_2) \in B \times B \mid W_1 \cap W_2 :\ e\text{-dimensional subspace not in } H\} \\
B_3 &:= \{(W_1, W_2) \in B \times B \mid W_1 \cap W_2 = U\}
\end{aligned}$$

For two G_U-orbitals K and K' and for $(W_1, W_2) \in K$, put

$$p(K, K') := \{W \in A \cup B \mid (W_1, W) \in K',\ W \text{ is adjacent to } W_2\}.$$

Then the following holds:

$p(K,K')$	A_0	A_1	A_2	A_3	A_4	AB_1	AB_2
A_0	0	$q-1$	$q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$	$q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}$	0	q^e	0
A_1	1	$q-2$	$q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$	$q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}$	0	q^e	0
A_2	1	$q-1$	$q^2\begin{bmatrix} e \\ 1 \end{bmatrix}-1$	0	$q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}$	0	q^e
A_3	1	$q-1$	0	$q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}-1$	$q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$	q^e	0
A_4	0	0	q	q	x	0	q^e
AB_1	1	$q-1$	0	$q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}$	0	q^e-1	$q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$
AB_2	0	0	q	0	$q^2\begin{bmatrix} e-2 \\ 1 \end{bmatrix}$	q	$q^2\begin{bmatrix} e \\ 1 \end{bmatrix}+q^e-q-1$

where $x = q^2\left(\begin{bmatrix} e \\ 1 \end{bmatrix} + \begin{bmatrix} e-2 \\ 1 \end{bmatrix}\right) - q - 1$. Considering the above array as a 7×7 matrix, the eigenvalues are $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$, $-q-1$ and -1. Similarly, we have the following results:

$$\begin{array}{ccccccc}
 & B_0 & B_1 & B_2 & B_3 & BA_1 & BA_2 \\
B_0 & 0 & q^e-1 & q^2\begin{bmatrix} e \\ 1 \end{bmatrix} & 0 & q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} & 0 \\
B_1 & 1 & q^e-2 & q^2 & q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix} & q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} & 0 \\
B_2 & 1 & q-1 & q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}+q^2-1 & q(q^2-1)\begin{bmatrix} e-1 \\ 1 \end{bmatrix} & 0 & q\begin{bmatrix} e-1 \\ 1 \end{bmatrix} \\
B_3 & 0 & q & q(q+1) & y & 0 & q\begin{bmatrix} e-1 \\ 1 \end{bmatrix} \\
BA_1 & 1 & q^e-1 & 0 & 0 & q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}-1 & q^2\begin{bmatrix} e \\ 1 \end{bmatrix} \\
BA_2 & 0 & 0 & q & q^e-q & q & z
\end{array}$$

where $y = q^3\begin{bmatrix} e \\ 1 \end{bmatrix} + q^e - 2q - 1$, $z = q^2\left(\begin{bmatrix} e \\ 1 \end{bmatrix} + \begin{bmatrix} e-2 \\ 1 \end{bmatrix}\right) - 1$. Considering the above array as a 6×6 matrix, the eigenvalues are $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$ and $-q-1$. Therefore we have the following conclusion.

Proposition 4 *For the local graph of $\tilde{J}_q(2e+1, e)$ around $W \in \mathcal{B}_1$, the eigenvalues are $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$, $-q-1$ and -1.*

The number of 3-cycles in $\Gamma(U)$ is equal to $qx(qx+1)(2q^3x^2 + (q^4 - q^3 - q^2 - 3q)x + q^3 - q^2 + 2)$. Let m_1, m_2, m_3 and m_4 be the multiplicities of $q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$, $-q-1$ and -1 respectively. From them, we conclude that $m_1 = \begin{bmatrix} e-1 \\ 1 \end{bmatrix}, m_2 = q\begin{bmatrix} e+1 \\ 1 \end{bmatrix} - 1, m_3 = q^2\left(\begin{bmatrix} e+1 \\ 1 \end{bmatrix} - q^{e-1}\right)\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$, and $m_4 = \begin{bmatrix} e-1 \\ 1 \end{bmatrix}(q^{e+1} - 1)$.

2.2

Fix $U \in \mathcal{B}_2$. Then the neighbors of U in Γ consists of the following three G-invariant sets:

$$\begin{aligned}
C &:= \{W \in \mathcal{B}_2 \mid W \cap U = U \cap H\}, \\
D &:= \{W \in \mathcal{B}_2 \mid W \cap U \neq U \cap H,\ W \text{ is adjacent to } U\}, \\
E &:= \{W \in \mathcal{B}_1 \mid W \subset U\}.
\end{aligned}$$

These three set forms the G_U-orbits on the neighbors of U. The G_U-orbitals on C are following:

$$\begin{aligned}
C_0 &:= \{(W_1, W_1) \mid W_1 \in C\}, \\
C_1 &:= \{(W_1, W_2) \in C \times C \mid \langle W_1, U\rangle = \langle W_2, U\rangle, W_1 \neq W_2\}, \\
C_2 &:= \{(W_1, W_2) \in C \times C \mid \langle W_1, U\rangle \neq \langle W_2, U\rangle\}.
\end{aligned}$$

The G_U-orbitals on $C \times D$ are following:

$$\begin{aligned} CD_1 &:= \{(W_1, W_2) \in C \times D \mid \dim W_1 \cap W_2 = e\}, \\ CD_2 &:= \{(W_1, W_2) \in C \times D \mid \dim W_1 \cap W_2 = e-1\}. \end{aligned}$$

The G_U-orbitals on $D \times C$ are $DC_1 := (CD_1)^t$ and $DC_2 := (CD_2)^t$. The sets $C \times E$ and $E \times C$ form G_U-orbitals. Let $W_1 \in D$. Then since $W \cap U$ is an e-dimensional subspace distinct from $U \cap H$, $U_1 := W_1 \cap U \cap H$ is an $(e-1)$-dimensional subspace and for some vectors $u_0, u_0' \in H$ and $u_1 \notin H$, $U = \langle U_1, u_0, u_1 \rangle$ and $W_1 = \langle U_1, u_0', u_1 \rangle$. The $(G_U)_{W_1}$-orbitals on D are following:

$$\begin{aligned} D_0 &:= \{W_1\}, \\ D_1 &:= \{\langle U_1, u_1, u \rangle \mid u \in \langle u_0, u_0' \rangle\}, \\ D_2 &:= \{\langle U_1, u_1, u \rangle \mid u \in H \setminus \langle u_0, u_0' \rangle\}, \\ D_3 &:= \{\langle U_1, au_0 + u_1, u_0' \rangle \mid a \in \mathbf{F}_q^\times\}, \\ D_4 &:= \{\langle U_1, au_0 + u_1, u \rangle \mid a \in \mathbf{F}_q^\times, u \in \langle u_0, u_0' \rangle \setminus \{\langle u_0 \rangle, \langle u_0' \rangle\}\}, \\ D_5 &:= \{\langle U_1, au_0 + u_1, u \rangle \mid a \in \mathbf{F}_q^\times, u \in H \setminus \langle U_1, u_0, u_0' \rangle\}, \\ D_6 &:= \{\langle U_1', au + u_1, bu + u_0', cu + u_0 \rangle \mid a, b, c \in \mathbf{F}_q, U_1 = \langle U_1', u \rangle, \dim U_1' = e-2\}, \\ D_7 &:= \{\langle U_1', au + u_1, bu + u_0', v \rangle \mid a, b \in \mathbf{F}_q, U_1 = \langle U_1', u \rangle, \dim U_1' = e-2, \\ &\qquad\qquad v \in H \setminus \langle U_1, u_0, u_0' \rangle\}. \end{aligned}$$

The G_U-orbitals on $D \times E$ are $DE_1 := \{(W_1, W_1 \cap U \cap H) \mid W_1 \in D\}$ and its complement DE_2. $ED_1 := (DE_1)^t$ and $ED_2 := (DE_2)^t$ form the G_U-orbitals on $E \times D$. The G_U-orbitals on $E \times E$ are $E_0 := \{(W_1, W_1) \mid W_1 \in E\}$ and its complement E_1. Consider matrices whose entries are $p(K, K')$. First we can obtain the following table:

	C_0	C_1	C_2	CD_1	CD_2	$C \times E$
C_0	0	$q-2$	$q^e - q$	$q^2 \begin{bmatrix} e \\ 1 \end{bmatrix}$	0	$\begin{bmatrix} e \\ 1 \end{bmatrix}$
C_1	1	$q-3$	$q^e - q$	$q^2 \begin{bmatrix} e \\ 1 \end{bmatrix}$	0	$\begin{bmatrix} e \\ 1 \end{bmatrix}$
C_2	1	$q-2$	$q^e - q - 1$	0	$q^2 \begin{bmatrix} e \\ 1 \end{bmatrix}$	$\begin{bmatrix} e \\ 1 \end{bmatrix}$
CD_1	1	$q-2$	0	$q^2 \begin{bmatrix} e \\ 1 \end{bmatrix} - 1$	$q^2 \begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	1
CD_2	0	0	$q-1$	q	α	1
$C \times E$	1	$q-2$	$q^e - q$	q^2	$q^3 \begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	$q \begin{bmatrix} e-1 \\ 1 \end{bmatrix}$

where $\alpha = q^2 \left(\begin{bmatrix} e \\ 1 \end{bmatrix} + \begin{bmatrix} e-1 \\ 1 \end{bmatrix} \right) - q - 1$. Considering the above array as a 6×6 matrix, the eigenvalues are $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2 \begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$, -1 and the roots of $x^2 - \left(q^2 \begin{bmatrix} e \\ 1 \end{bmatrix} - q - 2 \right) x - q^3 \begin{bmatrix} e \\ 1 \end{bmatrix} + q + 1 = 0$. The eigenvalue -1 has multiplicity 2

We note that the equation $x^2 - \left(q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - q - 2\right)x - q^3\begin{bmatrix} e \\ 1 \end{bmatrix} + q + 1 = 0$ has no roots in $\mathbf{Q}[q]$. Next we obtain the following table:

	E_0	E_1	ED_1	ED_2	$E \times C$
E_0	0	$q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	$q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$	0	$q^e - 1$
E_1	1	$q\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$	0	$q^2\begin{bmatrix} e \\ 1 \end{bmatrix}$	$q^e - 1$
ED_1	1	0	$q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} + q^2 - 1$	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	$q - 1$
ED_1	0	1	q^2	$q^2(q+1)\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$	$q - 1$
$E \times C$	1	$q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	q^2	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	$q^e - 2$

Considering the above array as a 5×5 matrix, the eigenvalues are $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q\begin{bmatrix} e \\ 1 \end{bmatrix} - 1$, $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$, $-q-1$ and -1. Finally we have the following tables:
(i):

	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7
D_0	0	$q-1$	$q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	$q-1$	$(q-1)^2$	0	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	0
D_1	1	$q-2$	$q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	$q-1$	$(q-1)^2$	0	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	0
D_2	1	$q-1$	$q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$	0	0	$q(q-1)$	0	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$
D_3	1	$q-1$	0	$q-2$	$(q-1)^2$	$q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	0
D_4	1	$q-1$	0	$q-1$	q^2-2q	$q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	0
D_5	0	0	q	1	$q-1$	α	0	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$
D_6	1	$q-1$	0	$q-1$	$(q-1)^2$	0	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1$	$q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$
D_7	0	0	q	0	0	$q(q-1)$	q	β

where $\alpha = q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix} + q^2 - 2q - 1$ and $\beta = q^2(q+1)\begin{bmatrix} e-1 \\ 1 \end{bmatrix} - q - 1$. Put this table D_{11}.
(ii):

	DC_1	DC_2	DE_1	DE_2
D_0	$q-1$	0	1	0
D_1	$q-1$	0	1	0
D_2	0	$q-1$	1	0
D_3	$q-1$	0	1	0
D_4	$q-1$	0	1	0
D_5	0	$q-1$	1	0
D_6	$q-1$	0	0	1
D_7	0	$q-1$	0	1

Put this table D_{12}.

(iii):

	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7
DC_1	1	$q-1$	0	$q-1$	$(q-1)^2$	0	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	0
DC_2	0	0	q	0	0	$q(q-1)$	0	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$
DE_1	1	$q-1$	$q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$	$q-1$	$(q-1)^2$	$q^2(q^{e-1}-1)$	0	0
DE_2	0	0	0	0	0	0	q^2	$q^3\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$

Put this table D_{21}.

(iv):

	DC_1	DC_2	DE_1	DE_2
DC_1	$q-2$	q^e-q	1	$q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$
DC_2	$q-1$	q^e-q-1	1	$q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$
DE_1	$q-1$	q^e-q	0	$q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$
DE_2	$q-1$	q^e-q	1	$q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}-1$

Put this table D_{22}.

Let $Z = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ be a 12×12 matrix. Then the eigenvalues and their multiplicities are as follows:

Eigenvalue	Multiplicities
$q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix}-1$	1
$q\begin{bmatrix} e \\ 1 \end{bmatrix}-1$	1
$q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}-1$	2
$-q-1$	3
-1	3
θ_1, θ_2	1 (for each root)

where θ_1 and θ_2 are the roots of $x^2 - \left(q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - q - 2\right)x - q^3\begin{bmatrix} e \\ 1 \end{bmatrix} + q + 1 = 0$.

Proposition 5 *For the local graph of $\tilde{J}_q(2e+1, e)$ with respect to $W \in \mathcal{B}_2$, the eigenvalues are $q(q+1)\begin{bmatrix} e \\ 1 \end{bmatrix}-1$, $q\begin{bmatrix} e \\ 1 \end{bmatrix}-1$, $q^2\begin{bmatrix} e-1 \\ 1 \end{bmatrix}-1$, $-q-1$, -1 and the roots of $x^2 - \left(q^2\begin{bmatrix} e \\ 1 \end{bmatrix} - q - 2\right)x - q^3\begin{bmatrix} e \\ 1 \end{bmatrix} + q + 1 = 0$.*

The number of 3-cycles in $\Gamma(U)$ is equal to the sum of numbers obtained from (1) to (9), which is $qx(q^5x^3+q^3x^3+3q^4x^2-6q^3x^2+q^2x^2+5q^3x-6q^2x^qx+2)$. The multiplicities m_1, m_2, m_3, m_4 and m_5 satisfy that for any $i \geq 0$,

$$(q(q+1)x-1)^i+(qx-1)^i m_1+(qx-q-1)^i m_2+(-q-1)^i m_3+(-1)^i m_4+a_i m_5 = \mathrm{Tr}A^i \quad (1)$$

where a_i is defined by as follows: $a_0 = 2$, $a_1 = q^2x-q-2$, $a_i = (q^2x-q-2)a_{i-1}+(q^3x-q-1)a_{i-2}$ for $i \geq 2$, which means that $\theta_1^i+\theta_2^i = a_i$ for any i. From them, we can see that $m_1 = q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$, $m_2 = q^e$, $m_3 = q^2\begin{bmatrix} e \\ 1 \end{bmatrix}\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$, $m_4 = (q^{e+1}-1)\begin{bmatrix} e \\ 1 \end{bmatrix} - q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$, and $m_5 = q\begin{bmatrix} e-1 \\ 1 \end{bmatrix}$.

3 Thin and non-thin irreducible modules

Let Γ be a distance-regular graph with classic parameter (d, q, α, β). For a local graph $\Gamma(x)$, if $\lambda \neq b_0 - b_1 - 1$ is an eigenvalue of the local graph, there exists an eigenvector v of $E_1^* A_1 E_1^*$ whose eigenvalue is λ. Then Tv forms an irreducible T-module of endpoint 1. Moreover any irreducible T-module of endpoint 1 is Tv for some eigenvector v of $E_1^* A_1 E_1^*$. P. Terwilliger proved the following [4]:

Theorem 6 *In the above assumption, the irreducible module Tv is thin if and only if*

$$\lambda \in \left\{ \alpha \begin{bmatrix} d-1 \\ 1 \end{bmatrix} - 1, \beta - \alpha - 1, -q-1, -1 \right\}$$

As we noted, the Grassmann graph $J_q(2e+1, e)$ and the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$ have classic parameter $(e, q, q, \begin{bmatrix} e+1 \\ 1 \end{bmatrix})$. In these cases,

$$\alpha \begin{bmatrix} d-1 \\ 1 \end{bmatrix} - 1 = q \begin{bmatrix} e-1 \\ 1 \end{bmatrix} - 1, \beta - \alpha - 1 = q \begin{bmatrix} e \\ 1 \end{bmatrix} - 1.$$

Hence the above set in the Theorem is just the eigenvalues of the local graph of Grassmann graph except $b_0 - b_1 - 1 = q(q+1) \begin{bmatrix} e \\ 1 \end{bmatrix} - 1$.

For the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$, let $U \in \mathcal{B}_1$. Then, from results in the previous section, we can see that the Terwilliger algebra $T(U)$ has 4 irreducible modules of endpoint 1. Moreover the above theorem, all such modules are thin. On the other hand, let $U \in \mathcal{B}_2$. Then there are 6 irreducible $T(U)$-modules of endpoint 1. From results in the previous section, we can see that three of them are thin and the other are non-thin. Therefore, the twisted Grassmann graph is a counterexample of the conjecture of Terwilliger.

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